Deriving The Formula for Converting a Query in the normal Space to that in LSI Space

Diagonal matrix of Singular Values (K×n)

Recall that $\boxed{M = U \, S \, V^T}$

Term-doc Matrix (K×N)

Eigenvectors of MM^T → The Term-Term Correlation Matrix (K×K)

Eigenvectors of M^TM → doc-doc Correlation Matrix (N×N)

Fact 1: U and V are orthonormal matrices.

That is $\boxed{U^T U = I}$ $\boxed{V^T V = I}$ I is Identity Matrix

$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{bmatrix}$$

makes sense since U and V are made up of Eigenvectors — which are orthogonal unit vectors and thus their dot product will be 1 for the same vectors and zero for non-same pair.

If we have the query $\underline{Q}$ and its representation in LSI Space is D_Q, Then Q and D_Q must satisfy the relation

$$Q = U \, S \, D_Q^T$$

$$U^T Q = \underbrace{\boxed{U^T U}}_{I} \, S \, D_Q^T$$ — multiply by U^T on both Sides and use fact 1

$$S^{-1} U^T Q = (\underbrace{S^{-1} S}_{I} \, D_Q^T)$$ — multiply by S^{-1} on both Sides and by definition $S^{-1} S = I$

Fact 2: If S is a diagonal matrix S^{-1} will be a diagonal Matrix too

$$S^{-1} U^T Q = D_Q^T$$

$$\underbrace{(D_Q^T)^T}_{D_Q} = (S^{-1} U^T Q)^T$$

— taking Transpose on both sides
noting Transpose of Transpose gives original

By the way Transpose

A bit on Transpose operator

If A is $m \times n$ matrix $[a_{ij}]$

A^T is $n \times m$ matrix $[a_{ji}]$

(ij^{th} element of A^T is ji^{th} element of A).

Fact 3 $(A^T)^T = A$ (easy)

Fact 4 $S^T = S$ If S is a Symmetric Matrix (ie $s_{ij} = s_{ji}$)

Fact 5 a diagonal matrix is a Symmetric Matrix

$$\begin{bmatrix} \ddots & 0 \\ 0 & \ddots \end{bmatrix}$$

Fact 6 $(AB)^T = B^T A^T$

Continuing.

$$D_Q = (S^{-1} U^T Q)^T$$

fact 4 with $A = S^{-1} U^T$, $B = Q$

$$= Q^T (S^{-1} U^T)^T$$

$$= Q^T \underbrace{(U^T)^T}_{U} \underbrace{(S^{-1})^T}_{S^{-1} \text{ (using facts 2,4,5)}} = \boxed{Q^T U S^{-1}} \quad QED!!$$